

# The geometry of extended null supersymmetry in M-theory

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## Abstract

For supersymmetric spacetimes in eleven dimensions admitting a null Killing spinor, a set of explicit necessary and sufficient conditions for the existence of any number of arbitrary additional Killing spinors is derived. The necessary and sufficient conditions are comprised of algebraic relationships, linear in the spinorial components, between the spinorial components and their first derivatives, and the components of the spin connection and four-form. The integrability conditions for the Killing spinor equation are also analysed in detail, to determine which components of the field equations are implied by arbitrary additional supersymmetries and the four-form Bianchi identity. This provides a complete formalism for the systematic and exhaustive investigation of all spacetimes with extended null supersymmetry in eleven dimensions. The formalism is employed to show that the general bosonic solution of eleven dimensional supergravity admitting a  $G_2$  structure defined by four Killing spinors is either locally the direct product of  $\mathbb{R}^{1,3}$  with a seven-manifold of  $G_2$  holonomy, or locally the Freund-Rubin direct product of  $AdS_4$  with a seven-manifold of weak  $G_2$  holonomy. In addition, all supersymmetric spacetimes admitting a  $(G_2 \ltimes \mathbb{R}^7) \times \mathbb{R}^2$  structure are classified.

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# 1 Introduction

The problem of systematically classifying supersymmetric spacetimes in string and M-theory has received great attention in recent years. A particularly important contribution was that of [1], which first identified the usefulness of the concept of G-structure in this context. Since then, G-structures have been used for the classification of all (minimally) supersymmetric spacetimes in many different supergravities, and special supersymmetric spacetimes of particular physical interest (for example,  $AdS$  spacetimes) in string and M-theory; for example, [2]-[20]. This programme has already led to numerous successes; for example, to the discovery of an infinite class of Einstein Sasaki manifolds [21], together with their field theory duals [22], [23]; a family of 1/2 BPS excitations of  $AdS$  [24]; supersymmetric  $AdS$  black holes [25], [26]; and supersymmetric black rings [27]-[30]. In addition, a classification of all minimally supersymmetric solutions of eleven dimensional supergravity was given in [9], [10]. The systematic and general nature of this method makes it a natural formalism to extend to the classification of supersymmetric spacetimes with extended supersymmetry. Indeed, following the suggestion of [9], a systematic procedure for applying G-structure ideas to the classification of spacetimes with extended supersymmetry was first given in [31], and illustrated in the context of a seven dimensional supergravity. An obvious target for this refined G-structure formalism is eleven dimensional supergravity.

Spinors in eleven dimensions fall into two distinct categories, according to whether their associated vector is timelike or null. Equivalently, they may be distinguished by their isotropy group; a timelike spinor is stabilised by  $SU(5)$ , while a null spinor has isotropy group  $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ . Hence the existence of a timelike or null Killing spinor implies that the spacetime admits an  $SU(5)$  or  $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$  structure respectively<sup>1</sup>. The existence of additional Killing spinors implies that the structure group is reduced to some subgroup - the common isotropy group of all the Killing spinors. Thus supersymmetric spacetimes in eleven dimensions may be naturally split into two classes: those with structure groups embedding in  $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ , and those with structure groups embedding in  $SU(5)$ ; of course there is some overlap between these classes, as spacetimes can admit both timelike and null Killing spinors. A complete list of the possible structure groups arising as subgroups of  $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ , together with the spaces of spinors they fix, is given in [32].

In [10], necessary and sufficient conditions on the spin connection and the four-form for the existence of a single null Killing spinor were derived. This paper is the fourth and final in a series [32], [33], [34], building on and refining the work of [10], in which the G-structure methods of [31] are used to derive, given the existence of a single null Killing spinor, necessary and sufficient conditions for the existence of any number of arbitrary additional Killing spinors. In [33], the general solution of the Killing spinor equation admitting a  $Spin(7)$  structure was derived. In [34], spacetimes admitting structure groups with a compact factor acting non-trivially on eight

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<sup>1</sup>More precisely, a Killing spinor is either null everywhere or it is not. In the former case, the spacetime admits a globally defined  $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$  structure. In the latter case, there exists a point and hence a neighbourhood of that point where the Killing spinor is timelike, and where it thus defines a preferred local  $SU(5)$  structure.

dimensions were studied. The results of this series, together with those of [10], provide the geometrical ‘‘DNA’’ of all eleven dimensional supersymmetric spacetimes with structure groups embedding in  $(Spin(7) \times \mathbb{R}^8) \times \mathbb{R}$ . The main aim of this paper is to complete this project, and to provide a complete reference on extended null supersymmetry in M-theory.

Given the existence of a single null Killing spinor  $\epsilon$  in eleven dimensions, one may always choose the spacetime basis

$$ds^2 = 2e^+e^- + \delta_{ij}e^i e^j + (e^9)^2, \quad (1)$$

where  $i, j = 1, \dots, 8$ , so that the spinor satisfies the projections

$$\begin{aligned} \Gamma_{1234}\epsilon &= \Gamma_{3456}\epsilon = \Gamma_{5678}\epsilon = \Gamma_{1357}\epsilon = -\epsilon, \\ \Gamma^+\epsilon &= 0. \end{aligned} \quad (2)$$

The eight-manifold spanned by the  $e^i$  is referred to as the base. With this choice of null Killing spinor, the Spin(7) invariant four-form defined on the base is given by

$$\begin{aligned} -\phi &= e^{1234} + e^{1256} + e^{1278} + e^{3456} + e^{3478} + e^{5678} + e^{1357} \\ &\quad - e^{1368} - e^{1458} - e^{1467} - e^{2358} - e^{2367} - e^{2457} + e^{2468}. \end{aligned} \quad (3)$$

We adopt all the conventions of [10], which are used consistently throughout this series of papers. Our objective is to extract, from the Killing spinor equation, the necessary and sufficient conditions for the existence of an arbitrary additional Killing spinor. Then, multi-spinor ansätze for the Killing spinor equation consistent with any desired structure group can be made at will. Let us briefly discuss the method we use; full details are to be found in [31]-[34].

We construct a basis for the space of Majorana spinors in eleven dimensions by acting on the Killing spinor equation with a particular subset of the Clifford algebra. The essential feature we require of our spinorial basis is that it preserve manifest Spin(7) covariance. In other words, we decompose the space of Majorana spinors into modules of the Spin(7) factor of the structure group defined by the null Killing spinor, and choose an appropriate basis for each module. Then, by likewise decomposing the spin connection and four-form into modules of Spin(7), we will see that we may convert the Killing spinor equation for an arbitrary additional Killing spinor into a set of purely bosonic equations for the Spin(7) tensors defining the Killing spinor. Specifically, we choose the basis to be

$$\epsilon, \Gamma^i\epsilon, \frac{1}{8}J_{ij}^A\Gamma^{ij}\epsilon, \Gamma^-\epsilon, \Gamma^{-i}\epsilon, \frac{1}{8}J_{ij}^A\Gamma^{-ij}\epsilon, \quad (4)$$

where the  $J^A$ ,  $A = 1, \dots, 7$ , are a set of two-forms defined on the base, which furnish a basis for the **7** of Spin(7), and we recognise the decomposition **32**  $\rightarrow$  **1 + 8 + 7 + 1' + 8' + 7'** of the space of Majorana spinors in eleven dimensions under Spin(7). We choose the explicit representation

of the  $J^A$  to be

$$\begin{aligned} J^1 &= e^{18} + e^{27} - e^{36} - e^{45}, & J^2 &= e^{28} - e^{17} - e^{35} + e^{46}, \\ J^3 &= e^{38} + e^{47} + e^{16} + e^{25}, & J^4 &= e^{48} - e^{37} + e^{15} - e^{26} \\ J^5 &= e^{58} + e^{67} - e^{14} - e^{23}, & J^6 &= e^{68} - e^{57} - e^{13} + e^{24}, \\ J^7 &= e^{78} + e^{56} + e^{34} + e^{12}. \end{aligned} \quad (5)$$

The  $J^A$  obey

$$J_{ik}^A J^{Bk}{}_j = -\delta^{AB} \delta_{ij} + K_{ij}^{AB}, \quad (6)$$

where the  $K_{ij}^{AB}$  are antisymmetric on  $AB$  and  $ij$ , and furnish a basis for the **21** of Spin(7).

Now, any additional Killing spinor  $\eta$  may be written as

$$\eta = (f + u_i \Gamma^i + \frac{1}{8} f^A J_{ij}^A \Gamma^{ij} + g \Gamma^- + v_i \Gamma^{-i} + \frac{1}{8} g^A J_{ij}^A \Gamma^{-ij}) \epsilon, \quad (7)$$

for thirty-two real functions  $f, u_i, f^A, g, v_i, g^A$ . We may always simplify some of the additional Killing spinors, by acting on them with the  $(Spin(7) \times \mathbb{R}^8) \times \mathbb{R}$  isotropy group of  $\epsilon$ . Since  $\epsilon$  is Killing,  $\eta$  is Killing if and only if

$$[\mathcal{D}_\mu, f + u_i \Gamma^i + \frac{1}{8} f^A J_{ij}^A \Gamma^{ij} + g \Gamma^- + v_i \Gamma^{-i} + \frac{1}{8} g^A J_{ij}^A \Gamma^{-ij}] \epsilon = 0, \quad (8)$$

where

$$\mathcal{D}_\mu = \partial_\mu + \frac{1}{4} \omega_{\mu\nu\sigma} \Gamma^{\nu\sigma} + \frac{1}{288} (\Gamma_{\mu\nu\sigma\tau\rho} - 8g_{\mu\nu}\Gamma_{\sigma\tau\rho}) F^{\nu\sigma\tau\rho} \quad (9)$$

is the supercovariant derivative. By imposing the projections satisfied by  $\epsilon$ , every spacetime component of (8) may be reduced to a manifest sum of basis spinors; then, the coefficient of each must vanish separately. Implementing this procedure produces a set of algebraic relations, linear in the functions defining the additional Killing spinor, between the functions, their first derivatives, and the components of the spin connection and fluxes.

While it is straightforward, the calculation of the commutator is extremely long and technical. The terms

$$[\mathcal{D}_\mu, f + \frac{1}{8} f^A J_{ij}^A \Gamma^{ij} + g \Gamma^- + \frac{1}{8} g^A J_{ij}^A \Gamma^{-ij}] \epsilon = 0 \quad (10)$$

have been calculated previously, in [33], [34]. Here we complete the calculation, and impose the  $N = 1$  constraints <sup>2</sup> on the result, so that it may be used directly for the analysis of extended supersymmetry. In quoting the result, we decompose the spin connection and four-form into irreducible Spin(7) modules; the reader is referred to [10] for full details of these decompositions.

Very recently, in [35], the action of the supercovariant derivative on a different basis of spinors was computed. In contrast to the present treatment, the authors of [35] did not assume the

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<sup>2</sup>Here, and throughout,  $N$  denotes the number of real linearly independent solutions of the  $d = 11$  Killing spinor equation.

existence of a single Killing spinor to begin with, and so do not impose the conditions for  $N = 1$  supersymmetry on their result. While this has the advantage that their results can in principle be used to analyse both extended timelike and null supersymmetry, it has the disadvantage that it does not exploit the drastic simplification arising from imposing the  $N = 1$  conditions in either case, and may not be used as easily for the study of extended supersymmetry.

This paper is organised as follows. In section two, we give the full set of equations obtained from calculating the commutator (8) for a general additional Killing spinor. We examine the integrability conditions for the Killing spinor equation to determine which components of the field equations are implied by the Bianchi identity and the existence of an arbitrary additional Killing spinor.

In section three, we use and illustrate the machinery of section two, to derive the general local bosonic solution of eleven dimensional supergravity, admitting a  $G_2$  structure defined by four Killing spinors. It is shown that the existence of an  $N = 4$   $G_2$  structure implies that spacetime is either locally the direct product of  $\mathbb{R}^{1,3}$  with a manifold of  $G_2$  holonomy, or locally the direct product of  $AdS_4$  with a manifold of weak  $G_2$  holonomy. This is derived without making any ansatz at all for the bosonic fields, and demanding only that the Killing spinors span a subspace of the space of spinors fixed by a  $G_2$  subgroup of  $\text{Spin}(1,10)$ . In section four, all supersymmetric spacetimes admitting a  $(G_2 \times \mathbb{R}^7) \times \mathbb{R}^2$  structure are classified. Section four concludes, and some technical material is relegated to the appendix.

## 2 Extended null supersymmetry in eleven dimensions

In this section, we will give the complete equations for arbitrary extended supersymmetry of a background in M-theory admitting a null Killing spinor. As discussed in the introduction, these equations are obtained by imposing

$$[\mathcal{D}_\mu, f + u_i \Gamma^i + \frac{1}{8} f^A J_{ij}^A \Gamma^{ij} + g \Gamma^- + v_i \Gamma^{-i} + \frac{1}{8} g^A J_{ij}^A \Gamma^{-ij}] \epsilon = 0. \quad (11)$$

We also impose the  $N = 1$  conditions on the spin connection and fluxes on the result we obtain, so that it may be used directly for the analysis of spacetimes with extended supersymmetry. The  $N = 1$  conditions associated with the existence of the Killing spinor  $\epsilon$  were derived in [10]. In order to make this section more self-contained, we will quote the result of these authors. The constraints on the spin connection are

$$\begin{aligned} \omega_{(\mu\nu)-} &= \omega_{ij-}^7 = \omega_{-ij}^7 = \omega_{i9-} = \omega_{-9i} = 0, \\ \omega_{+9-} &= \frac{1}{4} \omega_{i9}^i, \\ \omega_{9ij}^7 &= -\omega_{ij9}^7, \\ (\omega_{[ijk]}^7)^8 &= \frac{1}{56} \phi_{ijk}^l (\omega_{99l} - 6\omega_{l-+}). \end{aligned} \quad (12)$$

The conditions on the four-form are

$$\begin{aligned}
F_{+-9i} &= 2\omega_{i-+} - \omega_{99i}, \\
F_{+-ij} &= 2\omega_{[ij]9}, \\
F_{+9ij}^{\mathbf{7}} &= 2\omega_{+ij}^{\mathbf{7}}, \\
F_{+ijk}^{\mathbf{8}} &= \frac{2}{7}\phi_{ijk}{}^l\omega_{+9l}, \\
F_{-9ij}^{\mathbf{7}} &= 0, \\
F_{-9ij}^{\mathbf{21}} &= 2\omega_{ij-}^{\mathbf{21}}, \\
F_{-ijk} &= 0, \\
F_{9ijk}^{\mathbf{8}} &= \frac{2}{7}\phi_{ijk}{}^l(\omega_{99l} + \omega_{l-+}), \\
F_{9ijk}^{\mathbf{48}} &= -12(\omega_{[ijk]}^{\mathbf{7}})^{\mathbf{48}}, \\
F_{ijkl}^{\mathbf{1}} &= \frac{3}{7}\omega_{+9-}\phi_{ijkl}, \\
F_{ijkl}^{\mathbf{7}} &= 2\phi_{[ijk}{}^m\omega_{l]m9}^{\mathbf{7}}, \\
F_{ijkl}^{\mathbf{35}} &= 2\phi_{[ijk}{}^m\omega_{l]m9}^{\mathbf{35}}. \tag{13}
\end{aligned}$$

Here superscripts refer to irreducible Spin(7) modules;  $\omega_{ij-}^{\mathbf{7}}$ ,  $\omega_{ij9}^{\mathbf{7}}$  denote the  $\mathbf{7}$  projections of  $\omega_{[ij]-}$ ,  $\omega_{[ij]9}$  respectively, and  $\omega_{ij-}^{\mathbf{21}}$  denotes the  $\mathbf{21}$  projection of  $\omega_{[ij]-}$ . Here and henceforth,  $\omega_{ijk}^{\mathbf{7,21}}$  will denote the  $\mathbf{7,21}$  projections of  $\omega_{ijk}$  on  $j, k$ .  $(\omega_{[ijk]}^{\mathbf{7}})^{\mathbf{8,48}}$  denote the  $\mathbf{8,48}$  pieces of the completely antisymmetric part of  $\omega_{ijk}^{\mathbf{7}}$ . Finally,  $\omega_{ij9}^{\mathbf{35}} = \omega_{(ij)9} - \frac{1}{8}\delta_{ij}\omega_{k9}^k$ . The  $F_{+ijk}^{\mathbf{48}}$ ,  $F_{+9ij}^{\mathbf{21}}$  and  $F_{ijkl}^{\mathbf{27}}$  components of the four-form drop out of the Killing spinor equation for  $\epsilon$  and are unconstrained by the  $N = 1$  constraints. Now, given these constraints, we give the complete set of conditions for extended supersymmetry.

## 2.1 The complete conditions for extended supersymmetry

As we have discussed, the equations for extended supersymmetry are derived by imposing (11). Wherever possible, we have eliminated the four-form in favour of the spin connection, using the  $N = 1$  conditions. This is to maximise the ease with which the conditions for extended supersymmetry may be used; once the additional conditions on the spin connection have been solved, the components of the four-form fixed by the  $N = 1$  constraints may be read off from (13). Essentially, this procedure produces a set of linear first order partial differential equations for the functions defining the additional Killing spinors, which are satisfied by all spacetimes with extended supersymmetry in eleven dimensions which admit a null Killing spinor; it extracts the linearly independent equations contained in the Killing spinor equation. These necessary and

sufficient conditions for arbitrary extended supersymmetry are as follows.

The differential equations for the function  $f$  are:

$$\partial_- f + g \left[ \omega_{+9-} - \omega_{-+9} \right] + v^i \left[ \omega_{i-+} - \omega_{-+i} \right] = 0. \quad (14)$$

$$\partial_+ f - g\omega_{++9} - v^i \omega_{++i} = 0. \quad (15)$$

$$\partial_9 f + \frac{1}{3} u^i \left[ \omega_{99i} - 2\omega_{i-+} \right] + \frac{1}{3} f^A \omega_{ij9} J^{Aij} + g\omega_{99+} + v^i \left[ \omega_{9i+} - \frac{1}{3} \omega_{+9i} \right] - \frac{1}{3} g^A \omega_{+ij} J^{Aij} = 0. \quad (16)$$

$$\begin{aligned} & \partial_i f + u^j \left[ \frac{1}{2} \delta_{ij} \omega_{+9-} + \frac{2}{3} \omega_{ij9}^7 - \frac{2}{3} \omega_{ij9}^{21} - \frac{1}{3} \omega_{ij9}^{35} \right] - f^A \left[ \frac{4}{21} (\omega_{j-+} + \omega_{99j}) J^{Aj}{}_i + (\omega_{[ijk]}^7)^{48} J^{Ajk} \right] \\ & + g \left[ -\omega_{i+9} + \frac{1}{3} \omega_{+9i} \right] + v^j \left[ \omega_{ij+} - \frac{5}{3} \omega_{+ij}^7 - \frac{1}{6} F_{+9ij}^{21} \right] - g^A \left[ \frac{10}{21} \omega_{+9j} J^{Aj}{}_i + \frac{1}{12} F_{+ijk}^{48} J^{Ajk} \right] = 0. \end{aligned} \quad (17)$$

The differential equations for the  $u_i$  are:

$$\begin{aligned} & \partial_- u_i + u^j \left[ \omega_{-ij}^{21} + \frac{1}{3} \omega_{ij-}^{21} \right] + \frac{g}{3} \left[ 2\omega_{99i} - \omega_{i-+} - 3\omega_{-+i} \right] + v^j \left[ \omega_{-+9} \delta_{ij} - \frac{4}{3} \omega_{ij9}^7 \right. \\ & \left. + \frac{4}{3} \omega_{ij9}^{21} - \frac{2}{3} \omega_{ij9}^{35} \right] + g^A \left[ (-\omega_{-+j} - \frac{5}{7} \omega_{j-+} + \frac{2}{7} \omega_{99j}) J^{Aj}{}_i - 2(\omega_{[ijk]}^7)^{48} J^{Ajk} \right] = 0. \end{aligned} \quad (18)$$

$$\begin{aligned} & \partial_+ u_i + u^j \left[ -2\omega_{+ij}^7 + \omega_{+ij}^{21} + \frac{1}{2} F_{+9ij}^{21} \right] + f^A \left[ -\frac{4}{7} \omega_{+9j} J^{Aj}{}_i + \frac{1}{4} F_{+ijk}^{48} J^{Ajk} \right] - g\omega_{++i} + v_i \omega_{++9} \\ & - g^A \omega_{++j} J^{Aj}{}_i = 0. \end{aligned} \quad (19)$$

$$\begin{aligned} & \partial_9 u_i + u^j \left[ 2\omega_{ij9}^7 - \omega_{ij9}^{21} + \omega_{9ij}^{21} - \left( \frac{1}{2} \omega_{+9-} \delta_{ij} + \frac{2}{3} \omega_{ij9}^7 - \frac{2}{3} \omega_{ij9}^{21} - \frac{1}{3} \omega_{ij9}^{35} \right) \right] \\ & - f^A \left[ \frac{8}{21} (\omega_{j-+} + \omega_{99j}) J^{Aj}{}_i + 2(\omega_{[ijk]}^7)^{48} J^{Ajk} \right] - \frac{g}{3} \left[ 3\omega_{9+i} + \omega_{+9i} \right] \\ & + v^j \left[ -\omega_{99+} \delta_{ij} - \frac{4}{3} \omega_{+ij}^7 + \frac{2}{3} F_{+9ij}^{21} \right] - g^A \left[ (\omega_{9+j} - \frac{1}{21} \omega_{+9j}) J^{Aj}{}_i + \frac{1}{6} F_{+ijk}^{48} J^{Ajk} \right] = 0. \end{aligned} \quad (20)$$

$$\begin{aligned}
& \partial_i u_j + u^k \left[ \frac{8}{7} \delta_{k[i} \omega_{j]--} - (3\omega_{ijk}^{\mathbf{7}} - \omega_{ijk}^{\mathbf{21}}) + \frac{5}{14} \delta_{ik} (\omega_{99j} - 2\omega_{j-+}) + \frac{1}{7} \delta_{jk} (\omega_{99i} - 2\omega_{i-+}) \right. \\
& \left. - \frac{1}{6} \delta_{ij} (\omega_{99k} - 2\omega_{k-+}) + 4(\omega_{[ijk]}^{\mathbf{7}})^{\mathbf{48}} + \phi_{ij}^{lm} (\omega_{[klm]}^{\mathbf{7}})^{\mathbf{48}} \right] + f^A \left[ \frac{2}{7} \omega_{+9-} J_{ij}^A + \frac{1}{12} F_{ijkl}^{\mathbf{27}} J^{Ak} \right. \\
& \left. - \frac{1}{6} \delta_{ij} \omega_{kl9} J^{Ak} - \frac{2}{3} J^{Ak}{}_{[i} \omega_{j]k9}^{\mathbf{21}} + J^{Ak}{}_{(i} \omega_{j)k9}^{\mathbf{35}} \right] + g \left[ \omega_{ij+} - \frac{1}{3} \omega_{+ij}^{\mathbf{7}} + \frac{1}{2} F_{+9ij}^{\mathbf{21}} \right] + v^k \left[ (\omega_{i+9} \right. \\
& \left. + \frac{1}{21} \omega_{+9i}) \delta_{jk} - \frac{10}{21} \delta_{ik} \omega_{+9j} + \frac{2}{21} \delta_{ij} \omega_{+9k} - \frac{2}{21} \phi_{ijk}{}^l \omega_{+9l} + \frac{2}{3} F_{+ijk}^{\mathbf{48}} + \frac{1}{4} F_{+jlm}^{\mathbf{48}} \phi_{ik}{}^{lm} \right. \\
& \left. - \frac{1}{12} F_{+klm}^{\mathbf{48}} \phi_{ij}{}^{lm} \right] + g^A \left[ \frac{1}{6} \delta_{ij} \omega_{+kl} J^{Ak} + (\omega_{ik+} + \omega_{+ik}^{\mathbf{7}} + \frac{1}{6} F_{+9ik}^{\mathbf{21}}) J^{Ak}{}_j + \frac{1}{3} F_{+9jk}^{\mathbf{21}} J^{Ak}{}_i \right] = 0.
\end{aligned} \tag{21}$$

The differential equations for the  $f^A$  are:

$$\begin{aligned}
& \partial_- f^A + \frac{1}{4} f^B \left[ \omega_{-ij} - \frac{1}{3} \omega_{ij-} \right] K^{BAij} + \frac{2}{3} g \omega_{ij9} J^{Aij} + v^i \left[ \left( \frac{13}{21} \omega_{j-+} - \omega_{-+j} - \frac{8}{21} \omega_{99j} \right) J^{Aj}{}_i \right. \\
& \left. - 2(\omega_{[ijk]}^{\mathbf{7}})^{\mathbf{48}} J^{Ajk} \right] + g^B \left[ -(\omega_{-+9} + \frac{1}{7} \omega_{+9-}) \delta^{AB} - \frac{1}{24} J^{Aij} F_{ijkl}^{\mathbf{27}} J^{Bkl} + \frac{1}{3} \omega_{ij9} K^{BAij} \right] = 0.
\end{aligned} \tag{22}$$

$$\partial_+ f^A + u^i \left[ \frac{4}{7} \omega_{+9j} J^{Aj}{}_i - \frac{1}{4} F_{+ijk}^{\mathbf{48}} J^{Ajk} \right] + \frac{1}{4} f^B \left[ \omega_{+ij} - \frac{1}{2} F_{+9ij} \right] K^{BAij} - v^i \omega_{++j} J^{Aj}{}_i - g^A \omega_{++9} = 0. \tag{23}$$

$$\begin{aligned}
& \partial_9 f^A + u^i \left[ \left( \frac{5}{7} \omega_{99j} - \frac{2}{7} \omega_{j-+} \right) J^{Aj}{}_i + 2(\omega_{[ijk]}^{\mathbf{7}})^{\mathbf{48}} J^{Ajk} \right] + f^B \left[ \frac{1}{4} (\omega_{9ij} + \omega_{ij9}) K^{BAij} \right. \\
& \left. - \left( \frac{4}{7} \delta^{AB} \omega_{+9-} + \frac{1}{48} J^{Aij} F_{ijkl}^{\mathbf{27}} J^{Bkl} + \frac{1}{6} \omega_{ij9} K^{BAij} \right) \right] + \frac{1}{3} g \omega_{+ij} J^{Aij} - v^i \left[ (\omega_{9+j} - \frac{1}{21} \omega_{+9j}) J^{Aj}{}_i \right. \\
& \left. + \frac{1}{6} F_{+ijk}^{\mathbf{48}} J^{Ajk} \right] + g^B \left[ \delta^{AB} \omega_{99+} + \frac{1}{6} F_{+9ij} K^{BAij} \right] = 0.
\end{aligned} \tag{24}$$

$$\begin{aligned}
& \partial_i f^A + u^j \left[ -\frac{3}{28} \omega_{+9-} J_{ij}^A - \frac{1}{4} F_{ijkl}^{\mathbf{27}} J^{Ak}{}_{kl} + \frac{1}{2} \delta_{ij} J^{Ak}{}_{kl} \omega_{kl9}^{\mathbf{7}} + \omega_{jk9}^{\mathbf{7}} J^{Ak}{}_i - (\omega_{ik9}^{\mathbf{21}} + \frac{1}{2} \omega_{ik9}^{\mathbf{35}}) J^{Ak}{}_j \right. \\
& \left. + \left( \frac{1}{4} \delta_{jk} \omega_{+9-} - \frac{1}{3} \omega_{jk9}^{\mathbf{7}} + \frac{1}{3} \omega_{jk9}^{\mathbf{21}} - \frac{1}{6} \omega_{jk9}^{\mathbf{35}} \right) J^{Ak}{}_i \right] + f^B \left[ -\frac{1}{2} J_{ij}^{[B} J_{kl]}^A \omega^{jkl} + \frac{1}{4} (\omega_{ijk} K^{BAjk} \right. \\
& \left. - (2\omega_{j-+} + \omega_{99j}) K^{BAj}{}_i) + \left( \frac{4}{21} (\omega_{k-+} + \omega_{99k}) J^{Bk}{}_j + (\omega_{[jkl]}^{\mathbf{7}})^{\mathbf{48}} J^{Bkl} \right) J^{Aj}{}_i \right] + g \left[ \frac{2}{21} \omega_{+9j} J^{Aj}{}_i \right. \\
& \left. + \frac{1}{4} F_{+ijk}^{\mathbf{48}} J^{Ajk} \right] + v^j \left[ \frac{1}{6} \delta_{ij} \omega_{+kl}^{\mathbf{7}} J^{Ak}{}_{kl} + (\omega_{ik+} - \frac{1}{3} \omega_{+ik}^{\mathbf{7}} - \frac{1}{2} F_{+9ik}^{\mathbf{21}}) J^{Ak}{}_j + \frac{1}{3} F_{+9jk}^{\mathbf{21}} J^{Ak}{}_i \right] \\
& + g^B \left[ -\delta^{AB} (\omega_{i+9} + \frac{13}{126} \omega_{+9i}) - \frac{1}{18} \omega_{+9j} K^{BAj}{}_i - \frac{17}{144} F_{+ijk}^{\mathbf{48}} K^{BAjk} + \frac{1}{12} J_{ij}^{(A} J_{kl}^{B)} F_{+}^{\mathbf{48}jkl} \right] = 0.
\end{aligned} \tag{25}$$

The differential equations for  $g$  are:

$$\partial_- g = 0. \quad (26)$$

$$\partial_+ g + \frac{1}{3} u^i [\omega_{99i} - 2\omega_{i-+}] + \frac{1}{3} f^A \omega_{ij9} J^{Aij} - \frac{2}{3} v^i \omega_{+i9} - \frac{1}{3} g^A \omega_{+ij} J^{Aij} = 0. \quad (27)$$

$$\partial_9 g + v^i \omega_{99i} = 0. \quad (28)$$

$$\begin{aligned} & \partial_i g - \frac{2}{3} u^j \omega_{ij-}^{\mathbf{21}} + \frac{1}{3} g [2\omega_{i-+} - \omega_{99i}] - v^j [\delta_{ij} \omega_{+9-} + \frac{10}{3} \omega_{ij9}^7 + \frac{2}{3} \omega_{ij9}^{\mathbf{21}} + \frac{2}{3} \omega_{ij9}^{\mathbf{35}}] \\ & + g^A [\frac{1}{2} (\omega_{99j} J^{Aj}_i - \omega_{ijk} J^{Ajk}) - (\frac{1}{7} (\omega_{j-+} + \omega_{99j}) J^{Aj}_i - (\omega_{[ijk]}^7)^{\mathbf{48}} J^{Ajk})] = 0. \end{aligned} \quad (29)$$

The differential equations for the  $v^i$  are:

$$\partial_- v_i + v^j [\omega_{-ij}^{\mathbf{21}} - \omega_{ij-}^{\mathbf{21}}] = 0. \quad (30)$$

$$\begin{aligned} & \partial_+ v_i + u^j [\frac{1}{2} \delta_{ij} \omega_{+9-} + \frac{2}{3} \omega_{ij9}^7 - \frac{2}{3} \omega_{ij9}^{\mathbf{21}} - \frac{1}{3} \omega_{ij9}^{\mathbf{35}}] - f^A [\frac{4}{21} (\omega_{j-+} + \omega_{99j}) J^{Aj}_i + (\omega_{[ijk]}^7)^{\mathbf{48}} J^{Ajk}] \\ & - \frac{2}{3} g \omega_{+9i} + v^j [-\frac{2}{3} \omega_{+ij}^7 + \omega_{+ij}^{\mathbf{21}} - \frac{1}{6} F_{+9ij}^{\mathbf{21}}] - g^A [\frac{10}{21} \omega_{+9j} J^{Aj}_i + \frac{1}{12} F_{+ijk}^{\mathbf{48}} J^{Ajk}] = 0. \end{aligned} \quad (31)$$

$$\begin{aligned} & \partial_9 v_i + \frac{2}{3} u^j \omega_{ij-}^{\mathbf{21}} - \frac{2}{3} g [\omega_{99i} + \omega_{i-+}] + v^j [\frac{1}{2} \omega_{+9-} \delta_{ij} + \frac{4}{3} \omega_{ij9}^7 + \omega_{9ij}^{\mathbf{21}} - \frac{1}{3} \omega_{ij9}^{\mathbf{21}} - \frac{1}{3} \omega_{ij9}^{\mathbf{35}}] \\ & + g^A [-\frac{2}{7} (\omega_{j-+} + \omega_{99j}) J^{Aj}_i + 2(\omega_{[ijk]}^7)^{\mathbf{48}} J^{Ajk}] = 0. \end{aligned} \quad (32)$$

$$\begin{aligned} & \partial_i v_j + \frac{2}{3} f^A J^{Ak} {}_{[i} \omega_{j]k-}^{\mathbf{21}} + g [\frac{1}{2} \delta_{ij} \omega_{+9-} + \frac{4}{3} \omega_{ij9}^7 + \omega_{ij9}^{\mathbf{35}}] + v^k [\frac{8}{21} (\omega_{99[i} + \omega_{[i-+|}) \delta_{j]k} \\ & - \frac{1}{6} \phi_{ijk} {}^l (\omega_{99l} - 2\omega_{l-+}) + \omega_{ijk} - 4(\omega_{[ijk]}^7)^{\mathbf{48}} - \phi_{ij} {}^{lm} (\omega_{[lmk]}^7)^{\mathbf{48}}] + g^A [\frac{3}{14} \omega_{+9-} J_{ij}^A \\ & - \frac{1}{12} F_{ijkl}^{\mathbf{27}} J^{Ak} {}_{[i} \omega_{j]k9}^{\mathbf{21}} - J^{Ak} {}_{[i} \omega_{j]k9}^{\mathbf{35}}] = 0. \end{aligned} \quad (33)$$

The differential equations for the  $g^A$  are:

$$\partial_- g^A + \frac{1}{4} g^B [\omega_{-ij} + \omega_{ij-}] K^{BAij} = 0. \quad (34)$$

$$\begin{aligned} & \partial_+ g^A + u^i \left[ \left( \frac{1}{7} \omega_{99j} - \frac{6}{7} \omega_{j-+} \right) J^{Aj}_i - (\omega_{[ijk]}^7)^{\mathbf{48}} J^{Aj} \right] - f^B \left[ \frac{4}{7} \delta^{AB} \omega_{+9-} + \frac{1}{48} J^{Aij} F_{ijkl}^{\mathbf{27}} J^{Bkl} \right. \\ & \left. + \frac{1}{6} \omega_{ij9} K^{BAij} \right] + \frac{1}{3} g \omega_{+ij} J^{Aij} + v^i \left[ \frac{10}{21} \omega_{+9j} J^{Aj}_i + \frac{1}{12} F_{+ijk}^{\mathbf{48}} J^{Aj} \right] \\ & + \frac{1}{4} g^B \left[ \omega_{+ij} + \frac{1}{6} F_{+9ij} \right] K^{BAij} = 0. \end{aligned} \quad (35)$$

$$\begin{aligned} & \partial_9 g^A + \frac{1}{6} f^B \omega_{ij-} K^{BAij} - \frac{1}{3} g \omega_{ij9} J^{Aij} + v^i \left[ \left( \frac{13}{21} \omega_{99j} - \frac{8}{21} \omega_{j-+} \right) J^{Aj}_i - 2(\omega_{[ijk]}^7)^{\mathbf{48}} J^{Aj} \right] \\ & + g^B \left[ \frac{1}{4} (\omega_{9ij} + \omega_{ij9}) K^{BAij} + \left( \frac{4}{7} \delta^{AB} \omega_{+9-} + \frac{1}{48} J^{Aij} F_{ijkl}^{\mathbf{27}} J^{Bkl} - \frac{1}{6} \omega_{ij9} K^{BAij} \right) \right] = 0. \end{aligned} \quad (36)$$

$$\begin{aligned} & \partial_i g^A - u^j \left[ \omega_{ik-}^{\mathbf{21}} J^{Ak}_j - \frac{1}{3} \omega_{jk-}^{\mathbf{21}} J^{Ak}_i \right] + g \left[ -\frac{2}{21} (\omega_{99j} + \omega_{j-+}) J^{Aj}_i + 3(\omega_{[ijk]}^7)^{\mathbf{48}} J^{Aj} \right] \\ & + v^j \left[ -\frac{1}{7} \omega_{+9-} J_{ij}^A + \frac{1}{4} F_{ijkl}^{\mathbf{27}} J^{Akl} - \frac{1}{3} \delta_{ij} \omega_{kl9} J^{Akl} - \left( \frac{4}{3} \omega_{ik9}^7 + \omega_{ik9}^{\mathbf{21}} + \frac{1}{2} \omega_{ik9}^{\mathbf{35}} \right) J^{Ak}_j + \left( -\frac{2}{3} \omega_{jk9}^7 \right. \right. \\ & \left. \left. + \frac{1}{3} \omega_{jk9}^{\mathbf{21}} + \frac{1}{6} \omega_{jk9}^{\mathbf{35}} \right) J^{Ak} \right] + g^B \left[ \delta^{AB} \omega_{i-+} + \frac{1}{4} \left( \omega_{ijk} K^{BAjk} + (2\omega_{j-+} - \omega_{99j}) K^{BAj}_i \right) \right. \\ & \left. + \frac{1}{2} J_{ij}^{[B} J_{kl]}^A \omega^{jkl} + \left( \frac{1}{7} (\omega_{k-+} + \omega_{99k}) J^{Bk}_j - (\omega_{[jkl]}^7)^{\mathbf{48}} J^{Bkl} \right) J^{Aj}_i \right] = 0. \end{aligned} \quad (37)$$

## 2.2 Integrability conditions

The existence of a solution of the Killing spinor equation implies that some components of the field equations and Bianchi identity are identically satisfied. This follows from the (contracted) integrability condition for the Killing spinor equation. We will not undertake a complete analysis of the integrability condition here (though there it is entirely straightforward to do so, it comes at a significant computational cost); rather, we will assume that the Bianchi identity for the four-form,  $dF = 0$ , is always imposed on the solution of the Killing spinor equation. We will then use the integrability condition to determine which components of the field equations must

be imposed on the solution of the Killing spinor equation and Bianchi identity. The Einstein equation is

$$R_{\mu\nu} - \frac{1}{12} \left( F_{\mu\sigma\tau\rho} F_\nu^{\sigma\tau\rho} - \frac{1}{12} g_{\mu\nu} F_{\sigma\tau\rho\lambda} F^{\sigma\tau\rho\lambda} \right) = 0. \quad (38)$$

The (classical) four-form field equation is

$$\star \left( d \star F + \frac{1}{2} F \wedge F \right) = 0. \quad (39)$$

Note that in full M-theory both the field equations and the supersymmetry transformation receive higher-order corrections; while it is conceptually straightforward to incorporate these corrections into the formalism, here we restrict attention to classical supergravity. Given that the Bianchi identity is imposed, the contracted integrability condition for an arbitrary Killing spinor  $\eta$  reads

$$\Gamma^\nu [\mathcal{D}_\mu, \mathcal{D}_\nu] \eta = (E_{\mu\nu} \Gamma^\nu + Q^{\nu\sigma\tau} \Gamma_{\mu\nu\sigma\tau} - 6Q_{\mu\nu\sigma} \Gamma^{\nu\sigma}) \eta = 0, \quad (40)$$

where up to constant overall factors the Einstein and four-form field equations are respectively  $E_{\mu\nu} = 0$ ,  $Q_{\mu\nu\sigma} = 0$ . By taking  $\eta = \epsilon$ , and writing each spacetime component of (40) as a manifest sum of basis spinors, we find the following algebraic relationships between the components of the field equations:

$$\begin{aligned} E_{+-} &= E_{99} = 12Q_{+-9}, \\ E_{+i} &= 18Q_{+i9}, \\ E_{ij} &= -6Q_{+-9}\delta_{ij}. \end{aligned} \quad (41)$$

The components  $E_{++}$  and  $Q_{+ij}^{21}$  drop out of the integrability condition for  $\epsilon$  and are unconstrained. All other components of the field equations vanish identically. Thus, given the existence of the Killing spinor  $\epsilon$  and that the Bianchi identity is satisfied, it is sufficient to impose  $E_{++} = Q_{+9i} = Q_{+-9} = Q_{+ij}^{21} = 0$  to ensure that all field equations are satisfied.

Given the existence of a Killing spinor  $\epsilon$  and that the Bianchi identity is satisfied, the integrability condition for an arbitrary additional Killing spinor is

$$[\Gamma^\nu [\mathcal{D}_\mu, \mathcal{D}_\nu], f + u_i \Gamma^i + \frac{1}{8} f^A J_{ij}^A \Gamma^{ij} + g \Gamma^- + v_i \Gamma^{-i} + \frac{1}{8} g^A J_{ij}^A \Gamma^{-ij}] \epsilon = 0, \quad (42)$$

with  $\Gamma^\nu [\mathcal{D}_\mu, \mathcal{D}_\nu]$  given by (40). This equation is analysed in detail in the appendix, and here we will quote the result.

If in addition to  $\epsilon$  there exists a Killing spinor with  $v_i v^i \neq 0$ , then it is sufficient to impose the Bianchi identity to ensure that all field equations are satisfied.

If in addition to  $\epsilon$  there exists a Killing spinor with  $v_i = 0$ ,  $g^2 + g^A g^A \neq 0$ , then it is sufficient to impose the Bianchi identity and  $Q_{+-9} = 0$  to ensure that all field equations are satisfied.

If in addition to  $\epsilon$  there exists a Killing spinor with  $v_i = g = g^A = 0$ ,  $u_i u^i \neq 0$ , then it is sufficient to impose the Bianchi identity and  $E_{++} = Q_{+ij}^{21} = 0$  to ensure that all field equations are satisfied.

If in addition to  $\epsilon$  there exists a Killing spinor with  $v_i = g = g^A = u_i = 0$ ,  $f^A f^A \neq 0$ , then it is sufficient to impose the Bianchi identity and  $E_{++} = Q_{+-9} = Q_{+9i} = Q_{+ij}^{21} = 0$  to ensure that all field equations are satisfied.

### 3 The general solution of eleven dimensional supergravity admitting an $N = 4$ $G_2$ structure

In this section, we will illustrate how the full machinery of the previous section may be employed to perform exhaustive classifications of supersymmetric spacetimes. Specifically, we will derive the general local bosonic solution of eleven dimensional supergravity which admits four Killing spinors defining a  $G_2$  structure. As was shown in [32], additional Killing spinors defining a  $G_2$  structure may be taken to be of the form

$$(f + u_8 \Gamma^8 + g \Gamma^- + v_8 \Gamma^{-8})\epsilon. \quad (43)$$

We demand that in addition to  $\epsilon$ , there exist three linearly independent solutions of the Killing spinor of the form (43). We will thus classify all spacetimes with maximal  $G_2$  supersymmetry, since four is the greatest number of Killing spinors compatible with a  $G_2$  structure.

As was pointed out in [31], [34], supersymmetric spacetimes admitting the maximal number of Killing spinors consistent with a particular structure group are particularly easy to classify. This is because of the following argument. The conditions for supersymmetry given in the previous section are a set of  $11 \times 32$  partial differential equations, one for each spacetime component of the partial derivative of each spinorial component. For the case of maximal  $G_2$  supersymmetry, by demanding the existence of three additional Killing spinors of the specific form (43), we are assuming that the spinors span a four dimensional subspace of the space of spinors. Thus the full set of equations for each Killing spinor becomes a set of  $11 \times 4$  differential equations for the spinorial components, together with a set of  $11 \times 28$  algebraic equations for the spinorial components. Each of the algebraic equations for the spinorial components is of the schematic form

$$u_8 A + g B + v_8 C = 0. \quad (44)$$

The specific property of maximal  $G_2$  supersymmetry which greatly facilitates solving the Killing spinor equation, is that since we are demanding the existence of three additional linearly independent Killing spinors, in each of the algebraic equations for the spinorial components we must have  $A = B = C = 0$ . Similar arguments apply to maximal supersymmetry consistent with any other structure group. Let us now work through this procedure in detail.

### 3.1 The conditions for maximal $G_2$ supersymmetry

The  $G_2$  structure induces a  $1 + 7$  split of the tangent space of the eight dimensional base; we have chosen the frame so that the  $G_2$  structure group acts non-trivially on the  $e^A$ ,  $A = 1, \dots, 7$ . We have also chosen the  $J^A$ , given in the introduction, such that

$$\begin{aligned} J_{B8}^A &= \delta_{AB}, \\ J_{BC}^A &= -\Phi_{ABC}, \end{aligned} \quad (45)$$

where  $\Phi$  is the  $G_2$ -invariant associative three-form,

$$\Phi_{ABC} = \phi_{ABC8}. \quad (46)$$

Let us briefly discuss the decomposition of  $\text{Spin}(8)$  two-forms into irreducible modules of  $\text{Spin}(7)$  and  $G_2$ . Under  $\text{Spin}(7)$ , the **28** of  $\text{Spin}(8)$  decomposes as **28**  $\rightarrow$  **7+21**. Under  $G_2$ , the **7** of  $\text{Spin}(7)$  is irreducible, while the **21** decomposes as **21**  $\rightarrow$  **7' + 14**. We may effect the decomposition of a two-form  $\alpha$  in the **28** of  $\text{Spin}(8)$  into irreducible modules of  $\text{Spin}(7)$  using the  $J^A$  and  $K^{AB}$ :

$$\begin{aligned} \alpha_{ij}^{\mathbf{7}} &= \frac{1}{8} \alpha_{kl} J^{Akl} J_{ij}^A, \\ \alpha_{ij}^{\mathbf{21}} &= \frac{1}{16} \alpha_{kl} K^{ABkl} K_{ij}^{AB}. \end{aligned} \quad (47)$$

We may construct bases for the **7'** and the **14** by taking appropriate linear combinations of the  $K_{ij}^{AB}$ . These linear combinations are obtained by applying  $G_2$  projectors, as follows. A basis for the **7'** is given by

$$K_{ij}^{AB\mathbf{7'}} = \frac{1}{3} \left( K_{ij}^{AB} - \frac{1}{2} \Upsilon^{ABCD} K_{ij}^{CD} \right), \quad (48)$$

while a basis for the **14** is given by

$$K_{ij}^{AB\mathbf{14}} = \frac{2}{3} \left( K_{ij}^{AB} + \frac{1}{4} \Upsilon^{ABCD} K_{ij}^{CD} \right), \quad (49)$$

where  $\Upsilon$  is the  $G_2$  invariant coassociative four-form,

$$\Upsilon_{ABCD} = \phi_{ABCD}, \quad (50)$$

and  $\Upsilon = \star_7 \Phi$ , with orientation fixed by

$$\epsilon_{ABCDEFG} = \epsilon_{ABCDEFG8}. \quad (51)$$

For a two-form  $\beta$  in the **14** of  $G_2$ ,  $\Phi_A^{BC} \beta_{BC}^{\mathbf{14}} = 0$ . Then, since  $K_{A8}^{BC} = \Phi_{ABC}$ , we see from (47) that if

$$\alpha_{A8}^{\mathbf{21}} = 0, \quad (52)$$

then  $\alpha_{AB} = \alpha_{AB}^{14}$ .

Let us now solve the algebraic equations for the spinorial components. All three additional Killing spinors have  $u_A = f^A = v_A = g^A = 0$ , so by the argument given above, the coefficients of  $u_8$ ,  $g$  and  $v_8$  must vanish in all the differential equations for  $u_A$ ,  $f^A$ ,  $v_A$  and  $g^A$ . From the coefficient of  $g$  in (36), we find

$$\omega_{ij9}^7 = 0. \quad (53)$$

From the coefficient of  $v_8$  in the  $A$  component of (18),

$$\omega_{A89}^{21} = \frac{1}{2}\omega_{A89}^{35}, \quad (54)$$

and from the coefficient of  $u_8$  in the  $A$  component of (31),

$$\omega_{A89}^{21} = -\frac{1}{2}\omega_{A89}^{35}. \quad (55)$$

Hence

$$\omega_{A89}^{21} = \omega_{A89}^{35} = 0. \quad (56)$$

From the coefficient of  $u_8$  in the  $A$  component of (20),

$$\omega_{9A8}^{21} = 0. \quad (57)$$

From the coefficient of  $g$  in the  $iA$  component of (33),

$$\omega_{iA9}^{35} = -\frac{1}{2}\delta_{iA}\omega_{+9-}, \quad (58)$$

and hence, since  $\omega_{ij9}^{35}$  is traceless on  $i, j$ ,

$$\omega_{889}^{35} = \frac{7}{2}\omega_{+9-}. \quad (59)$$

From the coefficient of  $v_8$  in (37),

$$-\frac{1}{7}\omega_{+9-}\delta_{iA} + \frac{1}{4}F_{i8jk}^{27}J^{Aj} - \omega_{iA9}^{21} - \frac{1}{2}\omega_{iA9}^{35} - \frac{1}{6}\omega_{889}^{35}\delta_{iA} = 0. \quad (60)$$

The  $i = 8$  component of this equation produces nothing new. As in [34], we express  $F^{27}$  as

$$F^{27} = f^{AB} \left( J^A \wedge J^B - \frac{1}{7}\delta^{AB}J^C \wedge J^C \right), \quad (61)$$

so that

$$F_{B8jk}^{27}J^{Aj} = 8 \left( f^{(AB)} - \frac{1}{7}\delta^{AB}f^{CC} \right). \quad (62)$$

Then taking  $i = B$  in (60), from the antisymmetric part on  $A, B$  we find

$$\omega_{AB9}^{\mathbf{21}} = 0. \quad (63)$$

Tracing on  $A, B$  we find

$$\omega_{+9-} = \frac{14}{9}\omega_{889}^{\mathbf{35}}, \quad (64)$$

and hence  $\omega_{+9-} = \omega_{ij9}^{\mathbf{35}} = 0$ . Finally, the vanishing of the symmetric traceless part of (60) implies that

$$F_{ijkl}^{\mathbf{27}} = 0. \quad (65)$$

Using the  $N = 1$  conditions, we may summarise the additional constraints we have derived hitherto as

$$\begin{aligned} \omega_{+9-} = \omega_{ij9} &= \omega_{9A8} = F_{ijkl}^{\mathbf{27}} = 0, \\ \omega_{9AB} &= \omega_{9AB}^{\mathbf{14}}, \end{aligned} \quad (66)$$

where  $\omega_{9AB}^{\mathbf{14}}$  denotes the projection of  $\omega_{9AB}$  on the  $\mathbf{14}$ , or adjoint, of  $G_2$ .

Next, from the coefficient of  $u_8$  in (37),

$$\omega_{iA-}^{\mathbf{21}} - \frac{1}{3}\omega_{8j-}^{\mathbf{21}} J^{Aj}_i = 0. \quad (67)$$

Taking  $i = 8$  gives

$$\omega_{A8-}^{\mathbf{21}} = 0, \quad (68)$$

and then combined with the  $N = 1$  conditions,  $i = B$  produces

$$\omega_{ij-} = 0. \quad (69)$$

Then the  $A$  component of (30), together with the  $N = 1$  conditions, gives

$$\begin{aligned} \omega_{-A8} &= 0, \\ \omega_{-AB} &= \omega_{-AB}^{\mathbf{14}}. \end{aligned} \quad (70)$$

Next, from the coefficient of  $g$  in (25),

$$\frac{2}{21}\omega_{+9j}J^{Aj}_i + \frac{1}{4}F_{+ijk}^{\mathbf{48}}J^{Ajk} = 0. \quad (71)$$

From the coefficient of  $u_8$  in (23),

$$\frac{4}{7}\omega_{+9A} - \frac{1}{4}F_{+8jk}^{\mathbf{48}}J^{Ajk} = 0. \quad (72)$$

Comparing with the  $i = 8$  component of (71), we find

$$\omega_{+9A} = F_{+8jk}^{48} J^{Ajk} = 0. \quad (73)$$

Then the  $i = B$  component of (71) reads

$$-\frac{2}{21}\omega_{+98}\delta_{AB} - \frac{1}{4}F_{+BCD}^{48}\Phi^{ACD} - \frac{1}{2}F_{+8AB}^{48} = 0. \quad (74)$$

Tracing on  $A, B$ , and using the fact that for any three-form  $\alpha^{48}$  in the **48** of Spin(7),

$$\alpha_{ABC}^{48}\Phi^{ABC} = \alpha_{ijk}^{48}\phi^{ijk}_8 = 0, \quad (75)$$

we get

$$\omega_{+9i} = 0. \quad (76)$$

Then  $F_{+ijk}^{48}J^{Ajk} = 0$ , which implies that

$$F_{+ijk}^{48} = 0. \quad (77)$$

Then from the coefficient of  $v_8$  in (24), we get

$$\omega_{9+A} = 0. \quad (78)$$

Next, from the coefficient of  $g$  in (37),

$$-\frac{2}{21}(\omega_{99j} + \omega_{j-+})J^{Aj}_i + 3(\omega_{[ijk]}^7)^{48}J^{Ajk} = 0, \quad (79)$$

From the coefficient on  $v_8$  in (36),

$$\frac{13}{21}\omega_{99A} - \frac{8}{21}\omega_{A-+} - 2(\omega_{[8jk]}^7)^{48}J^{Ajk} = 0. \quad (80)$$

From the coefficient of  $u_8$  in (35),

$$\frac{1}{7}\omega_{99A} - \frac{6}{7}\omega_{A-+} - (\omega_{[8jk]}^7)^{48}J^{Ajk} = 0. \quad (81)$$

Combined with the  $i = 8$  component of (79), equations (80) and (81) imply that

$$\omega_{99A} = \omega_{A-+} = (\omega_{[8jk]}^7)^{48}J^{Ajk} = 0. \quad (82)$$

Then taking  $i = A$  in (79), we get

$$\omega_{998} = -\omega_{8-+}, \quad (83)$$

$$(\omega_{[ijk]}^7)^{48} = 0. \quad (84)$$

From the coefficient of  $v_8$  in (22),

$$\omega_{-+A} = 0. \quad (85)$$

From the coefficient of  $v_8$  in the  $iA$  component of (33),

$$\omega_{iA8} = 0. \quad (86)$$

From the coefficient of  $u_8$  in the  $iA$  component of (21),

$$3\omega_{iA8}^7 - \omega_{iA8}^{21} - \frac{1}{2}\delta_{iA}\omega_{8-+} = 0. \quad (87)$$

From (82) and (83), together with the  $N = 1$  conditions and the expression for the intrinsic contorsion of a Spin(7) structure given in [10], we may write

$$\omega_{ijk}^7 = \frac{1}{4}\delta_{i[j}\omega_{k]-+} - \frac{1}{8}\phi_{ijk}{}^l\omega_{l-+}. \quad (88)$$

Since  $\omega_{A-+} = 0$ ,  $\omega_{8jk}^7 = 0$ , (86) implies that  $\omega_{88A}^{21} = 0$ , and hence that

$$\omega_{8AB} = \omega_{8AB}^{14}. \quad (89)$$

Furthermore, (86) also implies that  $\omega_{AB8}^7 = -\omega_{AB8}^{21}$ , so (87) is implied by (86) and (88). Note that the  $i = B$  component of (87) may be rewritten as

$$\Phi_A{}^{CD}\omega_{BCD} = -\delta_{AB}\omega_{8-+}. \quad (90)$$

Next, from the coefficient of  $g$  in (35),

$$\omega_{+ij}^7 = 0. \quad (91)$$

From the coefficient of  $v_8$  in the  $A$  component of (20),

$$F_{+9A8}^{21} = 0. \quad (92)$$

Then from the coefficient of  $v_8$  in the  $A$  component of (31),

$$\begin{aligned} \omega_{+A8}^{21} &= 0, \\ \omega_{+AB} &= \omega_{+AB}^{14}. \end{aligned} \quad (93)$$

From the coefficient of  $v_8$  in (25),

$$\omega_{iA+} = \frac{1}{2}F_{+9iA}^{21}. \quad (94)$$

From the coefficient of  $g$  in the  $iA$  component of (21),

$$\omega_{iA+} = -\frac{1}{2}F_{+9iA}^{\mathbf{21}}. \quad (95)$$

Thus

$$F_{+9ij}^{\mathbf{21}} = \omega_{iA+} = 0. \quad (96)$$

Finally, from the coefficient of  $v_8$  in (23), we find

$$\omega_{++A} = 0. \quad (97)$$

**Summary** At this point we have solved all the algebraic equations for the spinorial components, so let us summarise what we have found. There are the following conditions on the spin connection:

$$\begin{aligned} \omega_{(\mu\nu)-} &= \omega_{i9-} = \omega_{-9i} = 0, \\ \omega_{+9-} &= \omega_{ij9} = \omega_{9A8} = 0, \\ \omega_{9AB} &= \omega_{9AB}^{\mathbf{14}}, \\ \omega_{ij-} &= \omega_{-A8} = 0, \\ \omega_{-AB} &= \omega_{-AB}^{\mathbf{14}}, \\ \omega_{+9i} &= \omega_{9+A} = 0, \\ \omega_{99A} = \omega_{A-+} &= \omega_{-+A} = 0, \\ \omega_{998} &= -\omega_{8-+}, \\ \omega_{88A} &= \omega_{AB8} = 0, \\ \omega_{8AB} &= \omega_{8AB}^{\mathbf{14}}, \\ \omega_{ijk}^7 &= \frac{1}{4}\delta_{i[j}\omega_{k]-+} - \frac{1}{8}\phi_{ijk}{}^l\omega_{l-+}, \\ \omega_{+A8} &= 0, \\ \omega_{+AB} &= \omega_{+AB}^{\mathbf{14}}, \\ \omega_{8A+} = \omega_{AB+} &= \omega_{++A} = 0. \end{aligned} \quad (98)$$

We have also found that the components of the flux not fixed by the  $N = 1$  constraints must vanish,  $F_{+ijk}^{\mathbf{48}} = F_{+9ij}^{\mathbf{21}} = F_{ijkl}^{\mathbf{27}} = 0$ . Inserting the additional conditions we have derived for the spin connection into the  $N = 1$  expressions for the flux, we find that the only nonzero component is

$$F_{+-98} = 3\omega_{8-+}. \quad (99)$$

It remains to solve the differential equations for  $f$ ,  $u_8$ ,  $g$  and  $v_8$ , for each of the four Killing spinors. Given the constraints we have derived on the spin connection, these equations reduce to

$$\partial_- f - g\omega_{-+9} + v_8 [\omega_{8-+} - \omega_{-+8}] = 0, \quad (100)$$

$$\partial_+ f - g\omega_{++9} - v_8\omega_{++8} = 0, \quad (101)$$

$$\partial_9 f - u_8\omega_{8-+} + g\omega_{99+} - v_8\omega_{9+8} = 0, \quad (102)$$

$$\partial_i f - g\omega_{i+9} + v_8\omega_{i8+} = 0. \quad (103)$$

$$\partial_- u_8 - g[\omega_{8-+} + \omega_{-+8}] + v_8\omega_{-+9} = 0, \quad (104)$$

$$\partial_+ u_8 - g\omega_{++8} + v_8\omega_{++9} = 0, \quad (105)$$

$$\partial_9 u_8 - g\omega_{9+8} - v_8\omega_{99+} = 0, \quad (106)$$

$$\partial_i u_8 - u_8\delta_{i8}\omega_{8-+} + g\omega_{i8+} + v_8\omega_{i+9} = 0. \quad (107)$$

$$\partial_- g = 0, \quad (108)$$

$$\partial_+ g - u_8\omega_{8-+} = 0, \quad (109)$$

$$\partial_9 g - v_8\omega_{8-+} = 0, \quad (110)$$

$$\partial_8 g + g\omega_{8-+} = 0, \quad (111)$$

$$\partial_A g = 0. \quad (112)$$

$$\partial_\mu v_8 = 0. \quad (113)$$

## 3.2 Solving the conditions for maximal $G_2$ supersymmetry

The only assumption we have made in deriving the above conditions is that there exist four linearly independent solutions of the Killing spinor equation stabilised by a common  $G_2$  subgroup of  $\text{Spin}(1,10)$ . Now we must solve the differential equations for the spinorial components, and also solve the conditions on the spin connection to determine the metric. To do so, we exploit the fact that we still have a lot of freedom left in our choice of spacetime basis. We are free to perform  $G_2$  rotations of the  $e^A$ , leaving all the Killing spinors invariant. We may also perform null rotations, about  $e^+$ , of the  $e^-$ ,  $e^9$  and  $e^8$ . These do not fix the individual Killing spinors, but they do preserve the four dimensional subspace spanned by the Killing spinors. These null rotations lie in an  $\mathbb{R}^2$  subgroup of the isotropy group,  $(\text{Spin}(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ , of  $\epsilon$ . A general element of this  $\mathbb{R}^2$ , in its spinorial representation, is

$$1 + p\Gamma^{+8} + q\Gamma^{+9}. \quad (114)$$

The action of the full  $(\text{Spin}(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$  on the spacetime basis is given in [10].

We will also exploit the fact that the differential equations for the spinorial components imply that all three additional Killing spinors must have  $v_8 = \text{constant}$ . Therefore, by taking linear combinations with constant coefficients, we may always arrange that only one of the additional Killing spinors has  $v_8 \neq 0$ . By assumption, at least one of the remaining pair has  $g \neq 0$ . Denote this spinor by  $\eta$ :

$$\eta = (f' + u'_8\Gamma^8 + g'\Gamma^-)\epsilon, \quad (115)$$

where  $g' \neq 0$  and without loss of generality we can always choose  $g' > 0$ . Now we can make a specific choice of frame, by acting on the Killing spinors with

$$1 + \frac{u'_8}{g'}\Gamma^{+8} + \frac{f'}{g'}\Gamma^{+9}, \quad (116)$$

so that in the new frame,

$$\eta = g'\Gamma^-\epsilon. \quad (117)$$

With this choice of frame, by examining the differential equations for the components of  $\eta$ , we find the additional conditions on the spin connection:

$$\begin{aligned} \omega_{-+9} = \omega_{++9} = \omega_{99+} = \omega_{i9+} &= \omega_{++8} = \omega_{9+8} = \omega_{i8+} = 0, \\ \omega_{8-+} &= -\omega_{-+8}, \end{aligned} \quad (118)$$

and the differential equations for  $g'$  are

$$\begin{aligned} \partial_- g' = \partial_+ g' &= \partial_9 g' = \partial_A g' = 0, \\ \partial_8 \log g' &= -\omega_{8-+}. \end{aligned} \quad (119)$$

Since  $\epsilon$  and  $\eta$  are stabilised by a common  $\text{Spin}(7)$  subgroup of  $\text{Spin}(1,10)$ , we have deduced that all spacetimes with maximal  $G_2$  supersymmetry also admit a  $\text{Spin}(7)$  structure. The conditions

on the spin connection associated with the existence of a Spin(7) structure were solved in [33]; the most general local metric may always be put in the form

$$ds^2 = (g'(x))^2 \left( 2[du + \lambda(x)_M dx^M][dv + \nu(x)_N dx^N] + [dz + \sigma(x)_M dx^M]^2 \right) \\ + (g'(x))^{-1} h_{MN}(x) dx^M dx^N, \quad (120)$$

where  $h_{MN}$  is a metric of Spin(7) holonomy and  $d\lambda$ ,  $d\nu$  and  $d\sigma$  are two-forms in the **21** of Spin(7). The spacetime basis is given by

$$\begin{aligned} e^+ &= g'^2(du + \lambda), \\ e^- &= dv + \nu, \\ e^9 &= g'(dz + \sigma), \\ e^i &= g'^{-1/2} \hat{e}^i(x)_M dx^M, \end{aligned} \quad (121)$$

where  $\hat{e}^i$  are the achtbeins for  $h$ . Thus our task reduces to imposing the additional constraints on the spin connection implied by the existence of the  $N = 4$   $G_2$  structure on the metric (120), and then determining the remaining Killing spinors. The spin connection for a metric of the form of (120) was calculated in [10]. It may be readily verified that having solved the conditions for a Spin(7) structure, the only remaining conditions on the spin connection for an  $N = 4$   $G_2$  that we have to solve are

$$\begin{aligned} \omega_{ij-} &= \omega_{ij+} = \omega_{ij9} = 0, \\ \omega_{88A} &= \omega_{AB8} = 0, \\ \omega_{8AB} &= \omega_{8AB}^{14}, \end{aligned} \quad (122)$$

together with the differential equations for the spinorial components.

Firstly, the conditions

$$\omega_{ij-} = \omega_{ij+} = \omega_{ij9} = 0 \quad (123)$$

imply that

$$d\lambda = d\nu = d\sigma = 0, \quad (124)$$

so by an  $x$ -dependent shift of the coordinates  $u, v, z$  we may always set  $\lambda = \nu = \sigma = 0$  locally. To proceed, let us introduce coordinates  $w, y^{\bar{A}}$  such that the vector  $\hat{e}^8$  is

$$\hat{e}^8 = \frac{\partial}{\partial w}, \quad (125)$$

and the achtbeins for  $h$  are

$$\begin{aligned} \hat{e}^8 &= (dw + \rho(w, y)_{\bar{A}} dy^{\bar{A}}), \\ \hat{e}^A &= \hat{e}_A^{\bar{A}}(w, y) dy^{\bar{A}}. \end{aligned} \quad (126)$$

There are two distinct cases to consider:  $\partial_8 g' = 0$ , and  $\partial_8 g' \neq 0$ .

**Case (i),**  $\partial_8 g' = 0$ . If  $\partial_8 g' = 0$ , then without loss of generality we may take  $g' = 1$ . We have  $\omega_{8-+} = 0$ , and  $e^i = \hat{e}^i$ . The condition  $\omega_{88A} = 0$  becomes

$$(\partial_w \rho)_A = 0, \quad (127)$$

so  $\rho = \rho(y)$ . Then  $\omega_{AB8}$  is given by

$$\omega_{AB8} = \Psi_{(AB)} + \frac{1}{2}d\rho_{AB}, \quad (128)$$

where

$$\Psi_{AB} = \delta_{AC}(\partial_w e^C)_B. \quad (129)$$

Hence

$$\Psi_{(AB)} = d\rho_{AB} = 0, \quad (130)$$

so locally we may set  $\rho = 0$ . Next,  $\omega_{8AB} = \omega_{8AB}^{14}$  implies that

$$\Psi_{AB} = \Psi_{AB}^{14}, \quad (131)$$

so that  $\Psi$  is a two-form in the adjoint of  $G_2$ . This means that the  $w$ -dependence of the  $e^A$  is pure gauge. Under a  $G_2$  rotation  $Q$ , the  $e^A$  transform according to

$$e^A \rightarrow (e^A)' = Q^A_B e^B. \quad (132)$$

By performing a  $w$ -dependent  $G_2$  rotation, we may choose the frame so that  $\Psi_{AB} = 0$ , while leaving all four Killing spinors invariant. Thus when  $g' = 1$ , the general local bosonic solution is the direct product of  $\mathbb{R}^{1,3}$  with a seven-manifold  $\mathcal{M}$ :

$$ds^2 = 2dudv + dz^2 + dw^2 + h_{\bar{A}\bar{B}}(y)dy^{\bar{A}}dy^{\bar{B}}. \quad (133)$$

Finally, requiring that the eight dimensional base space has  $\text{Spin}(7)$  holonomy implies that

$$\omega_{ABC} = \omega_{ABC}^{14}, \quad (134)$$

so  $\mathcal{M}$  must have  $G_2$  holonomy. The four-form vanishes, and the Einstein equations are identically satisfied. Finally, we may always choose the four Killing spinors to be

$$\epsilon, \Gamma^8 \epsilon, \Gamma^- \epsilon, \Gamma^- \epsilon. \quad (135)$$

**Case (ii):**  $\partial_8 g' \neq 0$ . Now suppose that  $\partial_8 g' \neq 0$ . In this case, it is convenient to convert the outstanding conditions on the spin connection from the  $e^i$  frame to the conformally rescaled  $\hat{e}^i$  frame. They become

$$\begin{aligned}\hat{\omega}_{88A} &= 0, \\ \hat{\omega}_{8AB} &= \hat{\omega}_{8AB}^{14}, \\ \hat{\omega}_{AB8} &= \frac{1}{2}\delta_{AB}\partial_8 \log g',\end{aligned}\tag{136}$$

and the condition that the conformally rescaled base has Spin(7) holonomy is

$$\hat{\omega}_{ijk}^7 = 0.\tag{137}$$

As before,  $\hat{\omega}_{88A} = \hat{\omega}_{[AB]8} = 0$  implies that locally we may set  $\rho = 0$ . Hence, from the differential equations for  $g'$ ,

$$g' = g'(w).\tag{138}$$

Again as before,  $\hat{\omega}_{8AB} = \hat{\omega}_{8AB}^{14}$  implies that  $\Psi_{[AB]}$  is pure gauge, and may be eliminated by a  $G_2$  rotation. Then  $\hat{\omega}_{AB8} = \frac{1}{2}\delta_{AB}\partial_8 g'$  implies that

$$\Psi_{AB} = \frac{1}{2}\delta_{AB}\partial_w \log g'.\tag{139}$$

This equation fixes the  $w$ -dependence of the  $\hat{e}^A$  to be

$$\hat{e}^A = g'^{1/2}(w)\tilde{e}^A(y),\tag{140}$$

so the eight-metric in the  $\hat{e}^8, \hat{e}^A$  frame becomes

$$ds^2 = dw^2 + g'(w)\tilde{h}_{\bar{A}\bar{B}}(y)dy^{\bar{A}}dy^{\bar{B}}.\tag{141}$$

The final condition we have to impose is that this is a metric of Spin(7) holonomy,  $\hat{\omega}_{ijk}^7 = 0$ . This condition is equivalent to

$$\delta_{AB}\partial_w \log g' = \hat{\Phi}_B^{CD}\hat{\omega}_{ACD},\tag{142}$$

where

$$\hat{\Phi} = g'^{3/2}\Phi.\tag{143}$$

By separating the  $w$ -dependence of both sides, there must exist some constant  $R^{-1}$  such that

$$\partial_w g' = R^{-1}g'^{1/2},\tag{144}$$

while in the conformally rescaled  $\hat{e}^A$  frame,

$$R^{-1}\delta_{AB} = \tilde{\Phi}_B^{CD}\tilde{\omega}_{ACD},\tag{145}$$

and

$$\tilde{\Phi} = \Phi = g'^{-3/2} \hat{\Phi}. \quad (146)$$

So

$$g' = \frac{1}{4} R^{-2} w^2, \quad (147)$$

and (145) is the statement that  $\tilde{h}_{\bar{A}\bar{B}}$  is a metric of weak  $G_2$  holonomy on the seven-manifold  $\mathcal{M}$  which is spanned by the  $\tilde{e}^A$ ; for more details on weak  $G_2$  manifolds, see, for example, [36]. This implies that the eleven dimensional metric is locally the direct product of  $AdS_4$  (with  $AdS$  length  $R$ ) with a seven manifold of weak  $G_2$  holonomy. Defining  $w = r^{-1/2}$ , and performing a constant rescaling of the coordinates  $u, v$  and  $z$ , the eleven-metric takes the form

$$ds^2 = \frac{R^2}{r^2} (2dudv + dz^2 + dr^2) + \tilde{h}_{\bar{A}\bar{B}}(y) dy^{\bar{A}} dy^{\bar{B}}. \quad (148)$$

The four-form is given by

$$F = -3R^{-1} e^+ \wedge e^- \wedge e^9 \wedge e^8. \quad (149)$$

The four Killing spinors are

$$\epsilon, \quad (4R^2 r)^{-1} \Gamma^-, \quad (-2vR^{-1} + zr^{-1} \Gamma^- + \Gamma^{-8})\epsilon, \quad (z + r\Gamma^8 + ur^{-1} \Gamma^-)\epsilon. \quad (150)$$

The Bianchi identity is identically satisfied. This is just the standard Freund-Rubin solution. Together with case (i), this provides the general local bosonic solution of eleven dimensional supergravity admitting four  $G_2$  invariant Killing spinors.

## 4 All spacetimes admitting a $(G_2 \ltimes \mathbb{R}^7) \times \mathbb{R}^2$ structure

In this section, all spacetimes admitting a  $(G_2 \ltimes \mathbb{R}^7) \times \mathbb{R}^2$  structure will be classified. As was shown in [32], all  $(G_2 \ltimes \mathbb{R}^7) \times \mathbb{R}^2$  structures in eleven dimensions are defined by a pair of Killing spinors; in addition to  $\epsilon$ , the second Killing spinor may always be taken to be

$$(f + u\Gamma^8)\epsilon. \quad (151)$$

We will now solve the equations of section two for this choice of additional Killing spinor.

### 4.1 The constraints for $(G_2 \ltimes \mathbb{R}^7) \times \mathbb{R}^2$ supersymmetry

With the above choice of Killing spinor, equations (22), (26), (28), (30), (33), (34) and (36) are identically satisfied. Equations (29) and (37) imply that

$$\begin{aligned} \omega_{i8-}^{21} &= 0, \\ \omega_{iA-}^{21} - \frac{1}{3} \omega_{8j-}^{21} J^{Aj}{}_i &= 0, \end{aligned} \quad (152)$$

and hence that

$$\omega_{ij-} = 0, \quad (153)$$

which from the  $N = 1$  conditions implies that

$$F_{-98A} = F_{-9AB} = 0. \quad (154)$$

Then equation (32) is satisfied, and the  $A$  component of (18) implies that

$$\begin{aligned} \omega_{-A8} &= 0, \\ \omega_{-AB} &= \omega_{-AB}^{\mathbf{14}}. \end{aligned} \quad (155)$$

Next, from the  $A$  component of (19), we find that

$$F_{+9A8}^{\mathbf{21}} = 4\omega_{+A8}^{\mathbf{7}} - 2\omega_{+A8}^{\mathbf{21}}. \quad (156)$$

The flux component  $F_{+9A8}$  is given by  $F_{+9A8} = F_{+9A8}^{\mathbf{7}} + F_{+9A8}^{\mathbf{21}}$ . From the  $N = 1$  condition on  $F_{+9ij}^{\mathbf{7}}$ , we find that

$$F_{+9A8} = 6\omega_{+A8}^{\mathbf{7}} - 2\omega_{+A8}^{\mathbf{21}} = -\omega_{+ij}\phi^{ij}_{-A8} = -\Phi_A{}^{BC}\omega_{+BC}. \quad (157)$$

Under  $G_2$ ,  $F_{+9AB}^{\mathbf{21}}$  contains both a  $\mathbf{7}$  and a  $\mathbf{14}$  part. The  $\mathbf{7}$  part is determined by (156). To see this, notice that

$$F_{+9A8}^{\mathbf{21}} = \frac{1}{2}\Phi_A{}^{BC}F_{+9BC}^{\mathbf{21}}. \quad (158)$$

Inverting this, we get

$$(F_{+9BC}^{\mathbf{21}})^{\mathbf{7G2}} = \frac{1}{3}F_{+9A8}^{\mathbf{21}}\Phi_A{}^{BC} = \frac{1}{3}(4\omega_{+A8}^{\mathbf{7}} - 2\omega_{+A8}^{\mathbf{21}})\Phi_A{}^{BC}, \quad (159)$$

where  $(F_{+9BC}^{\mathbf{21}})^{\mathbf{7G2}}$  denotes the projection of  $F_{+9BC}^{\mathbf{21}}$  on the  $\mathbf{7}$  of  $G_2$ . Similarly, for the flux components in the  $\mathbf{7}$  of  $\text{Spin}(7)$ , we find that

$$F_{+9BC}^{\mathbf{7}} = -F_{+9A8}^{\mathbf{7}}\Phi_A{}^{BC} = -2\omega_{+A8}^{\mathbf{7}}\Phi_A{}^{BC}. \quad (160)$$

Hence, the part of  $F_{+9BC}$  in the  $\mathbf{7}$  of  $G_2$  is given by

$$F_{+9BC}^{\mathbf{7G2}} = -\frac{2}{3}(\omega_{+A8}^{\mathbf{7}} + \omega_{+A8}^{\mathbf{21}})\Phi_A{}^{BC} = -\frac{2}{3}\omega_{+A8}\Phi_A{}^{BC}. \quad (161)$$

The components  $F_{+9AB}^{\mathbf{14}}$  in the  $\mathbf{14}$  of  $G_2$  drop out and are unconstrained.

Next consider equation (23). This gives

$$\frac{16}{7}\omega_{+9A} = F_{+8ij}^{\mathbf{48}}J^{Aij} = -F_{+8BC}^{\mathbf{48}}\Phi_A{}^{BC}, \quad (162)$$

or, equivalently,

$$(F_{+8BC}^{\mathbf{48}})^{\mathbf{7}_{\mathbf{G2}}} = -\frac{8}{21}\omega_{+9A}\Phi^A{}_{BC}. \quad (163)$$

Since from the  $N = 1$  conditions,  $F_{+8BC}^{\mathbf{8}} = -\frac{6}{21}\omega_{+9A}\Phi^A{}_{BC}$ , we obtain

$$F_{+8BC}^{\mathbf{7}_{\mathbf{G2}}} = -\frac{2}{3}\omega_{+9A}\Phi^A{}_{BC}. \quad (164)$$

Under  $G_2$ ,  $F_{+ABC} = F_{+ABC}^{\mathbf{8}+\mathbf{48}}$  contains  $\mathbf{1}$ ,  $\mathbf{7}$  and  $\mathbf{27}$  parts. We may extract the different pieces by contracting with  $\Phi_{ABC}$  and  $\Upsilon_{ABCD}$ . Since

$$\Phi^{ABC}F_{+ABC}^{\mathbf{48}} = \phi^{ijk}{}_8F_{+ijk}^{\mathbf{48}} = 0, \quad (165)$$

the singlet is given by

$$F_{+ABC}^{\mathbf{1}} = \frac{2}{7}\Phi_{ABC}\omega_{+98}. \quad (166)$$

Contracting  $F_{+ABC}^{\mathbf{8}}$  with  $\Upsilon$ , we get

$$F_{+ABC}^{\mathbf{8}}\Upsilon^{ABC}{}_D = \frac{48}{7}\omega_{+9D}. \quad (167)$$

We also find

$$F_{+ABC}^{\mathbf{48}}\Upsilon^{ABC}{}_D = F_{+ijk}^{\mathbf{48}}\phi^{ijk}{}_D + 3\Phi_D^{BC}F_{+8BC}^{\mathbf{48}} = -\frac{48}{7}\omega_{+9D}, \quad (168)$$

so

$$F_{+ABC}^{\mathbf{7}} = 0. \quad (169)$$

The components  $F_{+ABC}^{\mathbf{27}}$  drop out and are unconstrained.

Next, from the 8 component of (31), we find

$$\omega_{+9-} = \frac{2}{3}\omega_{889}^{\mathbf{35}}. \quad (170)$$

Since from the  $N = 1$  conditions  $\omega_{(ij)9} = \omega_{ij9}^{\mathbf{35}} + \frac{1}{2}\delta_{ij}\omega_{+9-}$ , this becomes

$$\omega_{889} = 2\omega_{+9-} = \omega^A{}_{A9}. \quad (171)$$

From the  $A$  component of (31),

$$2\omega_{A89}^{\mathbf{7}} - 2\omega_{A89}^{\mathbf{21}} - \omega_{A89}^{\mathbf{35}} = 0. \quad (172)$$

From the 8 component of (25), we obtain

$$3\omega_{A89}^{\mathbf{7}} + \omega_{A89}^{\mathbf{21}} - \frac{1}{2}\omega_{A89}^{\mathbf{35}} = 0. \quad (173)$$

The  $B$  component is

$$-\frac{3}{28}\delta_{AB}\omega_{+9-} + \frac{1}{4}F_{B8CD}^{\mathbf{27}_{\text{Spin}(7)}}\Phi^{ACD} + \omega_{8C9}^{\mathbf{7}}\Phi_{AB}^C + \omega_{AB9}^{\mathbf{21}} - \frac{1}{2}\omega_{AB9}^{\mathbf{35}} = 0. \quad (174)$$

The part antisymmetric on  $A, B$ , together with (172) and (173), implies that

$$\omega_{8A9} = \omega_{A89} = \omega_{[AB]9} = 0. \quad (175)$$

The  $N = 1$  conditions then imply that

$$F_{+-A8} = F_{+-AB} = \omega_{98A}^{\mathbf{7}} = 0. \quad (176)$$

The trace on  $A, B$  of (174) vanishes as a consequence of (170). Defining

$$\omega_{AB9}^{\mathbf{27}} = \omega_{(AB)9} - \frac{1}{7}\delta_{AB}\omega_{C9}^C, \quad (177)$$

the traceless symmetric part of (174) becomes

$$\frac{1}{2}F_{B8CD}^{\mathbf{27}_{\text{Spin}(7)}}\Phi^{ACD} - \omega_{AB9}^{\mathbf{27}} = 0. \quad (178)$$

Hence

$$F^{\mathbf{27}_{\text{Spin}(7)}} = -\frac{1}{4}\omega_{AB9}^{\mathbf{27}}(J^A \wedge J^B - \frac{1}{7}\delta^{AB}J^C \wedge J^C). \quad (179)$$

Then the  $A$  component of (20) implies that

$$\begin{aligned} \omega_{98A} &= 0, \\ \omega_{9AB} &= \omega_{9AB}^{\mathbf{14}}. \end{aligned} \quad (180)$$

It is now straightforward to verify that these conditions, together with the  $N = 1$  constraints, imply that  $F_{8ABC}$  and  $F_{ABCD}$  are determined as follows:

$$\begin{aligned} F_{8ABC}^{\mathbf{1}} &= -\frac{6}{7}\Phi_{ABC}\omega_{+9-}, \\ F_{8ABC}^{\mathbf{7}} &= 0, \\ F_{8ABC}^{\mathbf{27}_{\mathbf{G}_2}} &= -3\Phi_{[AB}^D\omega_{C]D9}^{\mathbf{27}}, \\ F_{ABCD} &= 0. \end{aligned} \quad (181)$$

Next, from (27), we find

$$\omega_{998} = 2\omega_{8-+}. \quad (182)$$

Equations (24) and (35) imply that

$$\begin{aligned}\omega_{99A} &= 2\omega_{A-+}, \\ ((\omega_{[8AB]}^7)^{48})^{7_{G2}} &= \frac{2}{21}\Phi_{AB}{}^C\omega_{C-+},\end{aligned}\tag{183}$$

where  $((\omega_{[8AB]}^7)^{48})^{7_{G2}}$  denotes the projection of  $(\omega_{[8AB]}^7)^{48}$  on the **7** of  $G_2$ . The  $8A$  component of (21) reads

$$\Phi_A{}^{BC}\omega_{8BC} = 0,\tag{184}$$

so that

$$\omega_{8AB} = \omega_{8AB}^{14}.\tag{185}$$

The  $AB$  component of (21) is

$$((\omega_{[8AB]}^7)^{48})^{14_{G2}} = -\frac{1}{12}\Phi_B{}^{CD}\omega_{ACD},\tag{186}$$

where  $((\omega_{[8AB]}^7)^{48})^{14_{G2}}$  denotes the projection of  $(\omega_{[8AB]}^7)^{48}$  on the **14** of  $G_2$ . Recall that for a  $G_2$  structure in seven dimensions, the intrinsic (con)torsion is specified by  $\omega_{ABC}^{7_{G2}}$ , and decomposes into four modules,

$$\begin{aligned}T \in \Lambda^1 \otimes g_2^\perp &= \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4, \\ \mathbf{7} \times \mathbf{7} &= \mathbf{1} + \mathbf{14} + \mathbf{27} + \mathbf{7}.\end{aligned}\tag{187}$$

Equation (186) states that  $\omega_{ABC}^{7_{G2}}$ , the projection on  $B, C$  of  $\omega_{ABC}$  onto the **7** of  $G_2$ , only contains a  $\mathcal{W}_2$  component.

At this point, we have solved all the purely algebraic equations contained in the Killing spinor equation. However, we do not yet have any explicit conditions on  $\omega_{88A}$ ,  $\omega_{AB8}$ . These are in fact contained in

$$\omega_{[ijk]}^7 = (\omega_{[ijk]}^7)^8 + (\omega_{[ijk]}^7)^{48}.\tag{188}$$

To extract them, we use the  $N = 1$  constraint on  $(\omega_{[ijk]}^7)^8$ , together with the conditions on  $(\omega_{[8AB]}^7)^{48}$ ,  $\omega_{99i}$ ,  $\omega_{i-+}$  and  $\omega_{ABC}$  derived above. The  $8AB$  component of (188) reduces to

$$\omega_{[AB]8} = \Phi_{AB}{}^C(\omega_{C-+} - \frac{1}{2}\omega_{88C}).\tag{189}$$

Observing that (186) implies that the totally antisymmetric part of  $\omega_{ABC}^{7_{G2}}$  vanishes,  $\omega_{[ABC]}^{7_{G2}} = 0$ , the  $ABC$  component of (188) becomes

$$-\frac{1}{4}\Phi_{[AB}{}^D\omega_{C]D8} = (\omega_{[ABC]}^7)^{48} - \frac{1}{14}\Phi_{ABC}\omega_{8-+} - \frac{1}{14}\Upsilon_{ABC}{}^D\omega_{D-+}.\tag{190}$$

The **1** part of this equation reads

$$\omega^A{}_{A8} = 2\omega_{8-+}. \quad (191)$$

The **7** part gives

$$\Phi_A{}^{BC}\omega_{BC8} = 0, \quad (192)$$

and hence

$$\begin{aligned} \omega_{[AB]8} &= 0, \\ \omega_{88C} &= 2\omega_{C-+}. \end{aligned} \quad (193)$$

The **27** part becomes

$$((\omega_{[ABC]}^7)^{48})^{27_{G2}} = -\frac{1}{4}\Phi_{[AB}{}^D\omega_{C]D9}^{27}, \quad (194)$$

where  $((\omega_{[ABC]}^7)^{48})^{27_{G2}}$  denotes the projection of  $(\omega_{[ABC]}^7)^{48}$  on the **27** of  $G_2$ , and  $\omega_{AB8}^{27}$  is defined in the same way as  $\omega_{AB9}^{27}$ . Now we may express the remaining flux components as follows:

$$\begin{aligned} F_{+-98} &= F_{+-9A} = 0, \\ F_{89AB}^7 &= 2\Phi_{AB}{}^C\omega_{C-+}, \\ F_{89AB}^{14} &= \Phi_A{}^{CD}\omega_{BCD}, \\ F_{9ABC}^1 &= \frac{6}{7}\Phi_{ABC}\omega_{8-+}, \\ F_{9ABC}^7 &= 0, \\ F_{9ABC}^{27} &= 3\Phi_{[AB}{}^D\omega_{C]D8}^{27}. \end{aligned} \quad (195)$$

Finally, it is easily verified that the algebraic conditions we have derived imply that the differential equations for the spinorial components reduce to

$$\begin{aligned} \partial_\mu f &= 0, \\ \partial_\mu u &= 0. \end{aligned} \quad (196)$$

Hence given a solution of the conditions on the spin connection and flux, we may always choose the second linearly independent Killing spinor to be

$$\Gamma^8\epsilon. \quad (197)$$

## 4.2 Summary

Let us now summarise the above conditions for  $(G_2 \ltimes \mathbb{R}^7) \times \mathbb{R}^2$  supersymmetry. In the spacetime basis

$$ds^2 = 2e^+e^- + \delta_{AB}e^Ae^B + (e^8)^2 + (e^9)^2, \quad (198)$$

the Killing spinors may be chosen to be

$$\epsilon, \Gamma^8\epsilon. \quad (199)$$

The components of the spin connection are required to satisfy

$$\begin{aligned} \omega_{(\mu\nu)-} &= \omega_{ij-} = \omega_{i9-} = \omega_{-9i} = \omega_{-A8} = 0, \\ \omega_{-AB} &= \omega_{-AB}^{\mathbf{14}}, \\ \omega_{889} &= 2\omega_{+9-} = \omega_{A9}^A, \\ \omega_{8A9} = \omega_{A89} &= \omega_{98A} = \omega_{[AB]9} = 0, \\ \omega_{9AB} &= \omega_{9AB}^{\mathbf{14}}, \\ \omega_{998} &= 2\omega_{8-+} = \omega_{A8}^A, \\ \omega_{99A} &= 2\omega_{A-+} = \omega_{88A}, \\ \omega_{[AB]8} &= 0, \\ \omega_{8AB} &= \omega_{8AB}^{\mathbf{14}}, \\ \Phi_A^{CD}\omega_{BCD}^{\mathbf{7}} &= (\Phi_{[A}^{CD}\omega_{B]CD}^{\mathbf{7}})^{\mathbf{14}}, \end{aligned} \quad (200)$$

where bold-face superscripts refer to  $G_2$  representations and  $\omega_{ABC}^{\mathbf{7}}$  denotes the  $\mathbf{7}$  projection of  $\omega_{ABC}$  on  $B, C$ . Given a geometry satisfying the above conditions, the only nonzero components of the flux are (again with bold face superscripts referring to  $G_2$  representations)

$$\begin{aligned} F_{+89A} &= -\Phi_A^{BC}\omega_{+BC}, \\ F_{+9AB}^{\mathbf{7}} &= -\frac{2}{3}\omega_{+C8}\Phi_{AB}^C, \\ F_{+8AB}^{\mathbf{7}} &= \frac{2}{3}\omega_{+C9}\Phi_{AB}^C, \\ F_{+ABC}^{\mathbf{1}} &= \frac{2}{7}\Phi_{ABC}\omega_{+98}, \\ F_{89AB}^{\mathbf{7}} &= 2\Phi_{AB}^C\omega_{C-+}, \\ F_{89AB}^{\mathbf{14}} &= \Phi_A^{CD}\omega_{BCD}, \\ F_{9ABC}^{\mathbf{1}} &= \frac{6}{7}\Phi_{ABC}\omega_{8-+}, \\ F_{9ABC}^{\mathbf{27}} &= 3\Phi_{[AB}^D\omega_{C]D8}^{\mathbf{27}}, \\ F_{8ABC}^{\mathbf{1}} &= -\frac{6}{7}\Phi_{ABC}\omega_{+9-}, \\ F_{8ABC}^{\mathbf{27}} &= -3\Phi_{[AB}^D\omega_{C]D9}^{\mathbf{27}}, \end{aligned} \quad (201)$$

together with  $F_{+9AB}^{14}$  and  $F_{+ABC}^{27}$  which drop out of the Killing spinor equations for  $\epsilon$  and  $\Gamma^8\epsilon$  and are unconstrained by supersymmetry. Given a solution of these conditions, it is sufficient to impose the Bianchi identity and  $E_{++} = Q_{+ij} = 0$  to ensure that all field equations are satisfied.

## 5 Conclusions

In this work, a formalism for the exhaustive investigation of all spacetimes with extended null supersymmetry in M-theory has been provided. There is clearly much scope for the exploration of the equations of section two. One way in which this could be systematically pursued is by working through all the possible structure groups of [32], case by case. In particular, it should now be straightforward to classify, reasonably explicitly, all spacetimes with maximal supersymmetry consistent with a given structure group  $G$ , as was done for  $G_2$  here. More challenging will be the classification of spacetimes with less than maximal  $G$ -supersymmetry, particularly for spacetimes with few supersymmetries and small structure groups, since in these cases the Killing spinors will be quite generic and will involve many functions, leading to a very complicated system of equations. Spacetimes admitting generic identity structures will be particularly difficult to cover in full generality. One would (perhaps naively) expect spacetimes with more than sixteen supersymmetries to be reasonably simple (indeed, it was shown in [37] that all spacetimes with more than twenty-four are locally homogeneous). However, such spacetimes necessarily admit an identity structure, and it is precisely this case which seems to be hardest to treat using  $G$ -structure methods. It would be interesting to get a proper handle on these cases.

In addition to the general classification programme, it will be interesting to use the formalism for highly targeted searches for solutions of particular physical interest. One can insert, into the equations of section two, any desired ansatz for the Killing spinors, the metric, or both; used in this way, the refined  $G$ -structure formalism of this paper is identical in spirit to a completely general version of the algebraic Killing spinor techniques of [38]. Of course, in any given context, the conditions for supersymmetry may be recalculated using other techniques, most notably algebraic Killing spinors; but the machinery of this paper covers *every* context where a null Killing spinor exists, and this generality of the formalism ensures the generality of its applications.

The usefulness of the refined  $G$ -structure formalism stems from the fact that it exploits the existence of a  $G$ -structure to extract and package the linearly independent first order PDEs contained within the Killing spinor equation in a compact and (reasonably) tractable form. It also provides a great deal of geometrical insight into the implications of supersymmetry, through the relationship between the fluxes and the intrinsic (con)torsion, for example. Nevertheless, one is still left with the task of solving these equations, and more seriously, with solving the Bianchi identity and some subset of the field equations. This will inevitably prove to be intractably difficult in many cases. However, the general overview of the Killing spinor equation provided by this approach should help reveal the directions in which real progress can be made.

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## A The integrability conditions

Here we will analyse the integrability conditions for an arbitrary additional Killing spinor  $\eta$ , assuming the existence of the Killing spinor  $\epsilon$ , and that the Bianchi identity for the four-form is imposed on the solution of the Killing spinor equation. The integrability condition we analyse is

$$[\Gamma^\nu [\mathcal{D}_\mu, \mathcal{D}_\nu], f + u_i \Gamma^i + \frac{1}{8} f^A J_{ij}^A \Gamma^{ij} + g \Gamma^- + v_i \Gamma^{-i} + \frac{1}{8} g^A J_{ij}^A \Gamma^{-ij}] \epsilon = 0, \quad (202)$$

where

$$\Gamma^\nu [\mathcal{D}_\mu, \mathcal{D}_\nu] = E_{\mu\nu} \Gamma^\nu + Q^{\nu\sigma\tau} \Gamma_{\mu\nu\sigma\tau} - 6Q_{\mu\nu\sigma} \Gamma^{\nu\sigma}. \quad (203)$$

The algebraic conditions on the components of the field equations implied by the existence of the Killing spinor  $\epsilon$  are given in subsection 2.2. To obtain the conditions implied by the existence of an arbitrary additional Killing spinor, we impose the projections satisfied by  $\epsilon$  to write each spacetime component of (202) as a manifest sum of basis spinors. The vanishing of each coefficient then gives the conditions on the components of the field equations. From the  $-$  component, we find

$$v_i Q_{+-9} = 0. \quad (204)$$

From the  $+$  component, we obtain

$$18Q_{+9i} u^i - gE_{++} = 0, \quad (205)$$

$$-18Q_{+ij}^{\mathbf{21}} u^j + E_{++} v_i = 0, \quad (206)$$

$$-36Q_{+9j} J_{i}^{Aj} u^i - 9Q_{+ij}^{\mathbf{21}} f^B K^{BAij} + 2E_{++} g^A = 0, \quad (207)$$

$$Q_{+9i} v^i = 0, \quad (208)$$

$$2Q_{+-9} u_i + 2gQ_{+9i} - Q_{+ij}^{\mathbf{21}} v^j + 2g^A Q_{+9j} J_{i}^{Aj} = 0, \quad (209)$$

$$4Q_{+9j} J_{i}^{Aj} v^i - Q_{+ij}^{\mathbf{21}} g^B K^{BAij} = 0. \quad (210)$$

The only independent condition obtained from the 9 component is

$$Q_{+-9} u_i + gQ_{+9i} + Q_{+ij}^{\mathbf{21}} v^j + Q_{+9j} g^A J_{i}^{Aj} = 0. \quad (211)$$

Finally from the  $i$  component, we get the new conditions

$$3gQ_{+ij}^{\mathbf{21}} + Q_{+9j} v_i + 5Q_{+9i} v_j + \phi_{ij}^{kl} Q_{+9k} v_l + g^A [2Q_{+jk}^{\mathbf{21}} J_{i}^{Ak} + Q_{+ik}^{\mathbf{21}} J_{j}^{Ak}] = 0, \quad (212)$$

$$\left[ Q_{+-9} u_j + g Q_{+9j} + Q_{+jk}^{\mathbf{21}} v^k + Q_{+9k} g^A J^{Ak} \right] J^{Aj}{}_i + 6v^j Q_{+k[i}^{\mathbf{21}} J^{Ak}] = 0. \quad (213)$$

Let us now analyse in detail the implications of these equations. Combining (209) and (211), we find that

$$Q_{+ij}^{\mathbf{21}} v^j = 0. \quad (214)$$

Then combining (211) and (213), we find that

$$Q_{+kj}^{\mathbf{21}} J^{Aj}{}_i v^i = 0. \quad (215)$$

Since  $v^j, J^{Aj}{}_i v^i$  are the components of eight linearly independent vectors, (214) and (215) imply that

$$v_i Q_{+jk}^{\mathbf{21}} = 0. \quad (216)$$

Then by considering the cases  $v_i = 0$  and  $v_i \neq 0$  separately, from (206) we find that

$$u^j Q_{+ij}^{\mathbf{21}} = v_i E_{++} = 0. \quad (217)$$

Similarly from (210),

$$Q_{+9j} J^{Aj}{}_i v^i = g^A Q_{+ij}^{\mathbf{21}} K^{ABij} = 0, \quad (218)$$

and then (208) yields

$$v_i Q_{+9j} = 0. \quad (219)$$

Then given (204) we see that if there exists an additional Killing spinor with  $v_i v^i \neq 0$ , it is sufficient to impose the Bianchi identity, and all components of the field equations are identically satisfied.

Now suppose that  $v_i = 0$ . Since  $Q_{+ij}$  only has components in the **21**, the coefficient of  $g^A$  in equation (212) only has components in the **7** and **35** (and in fact, the **7** part vanishes as a consequence of (218)). Hence

$$g Q_{+ij}^{\mathbf{21}} = 0. \quad (220)$$

Then the vanishing of the **35** part of the coefficient of  $g^A$  in (212), together with (218), implies that

$$g^A Q_{+ij}^{\mathbf{21}} = 0. \quad (221)$$

We have now solved all the integrability conditions with  $v_i = 0$ , save for equations (205), (207) and (209). Consider first the case  $g^2 + g^A g^A \neq 0$ . Then if we *impose*  $Q_{+-9} = 0$ , equation (209) becomes

$$g Q_{+9i} + g^A Q_{+9j} J^{Aj}{}_i = 0. \quad (222)$$

Contracting with  $Q_{+9}{}^i$  we find that  $g Q_{+9i} = 0$ , and contracting with  $J^{Bij} Q_{+9j}$  we find that  $g^A Q_{+9i} = 0$ . Hence

$$(g^2 + g^A g^A) Q_{+9i} = 0. \quad (223)$$

Since  $g^A Q_{+ij}^{\mathbf{21}} = 0$ , equations (205) and (207) reduce to

$$(g^2 + g^A g^A) E_{++} = 0. \quad (224)$$

So when there exists an additional Killing spinor with  $v_i = 0$ ,  $g^2 + g^A g^A \neq 0$ , it is sufficient to impose the Bianchi identity and  $Q_{+-9} = 0$  to ensure that all field equations are satisfied.

Finally consider the case  $v_i = g = g^A = 0$ . If there exists an additional Killing spinor with  $u_i u^i \neq 0$ , then from (205), (207) and (209) it is sufficient to impose  $E_{++} = Q_{+ij}^{\mathbf{21}} = 0$  in addition to the Bianchi identity. If there exists an additional Killing spinor with  $v_i = g = g^A = u_i = 0$ , then it is sufficient to impose  $E_{++} = Q_{+-9} = Q_{+9i} = Q_{+ij}^{\mathbf{21}} = 0$ .

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